LAST LECTURE

This talk will be a bit of a hodgepodge w/ no precise agenda.

My main aims are

① to remind you of what we've done in the course and

② to give a sense of how it connects to some current channels of research

It will be an impressionistic talk!

§0. Big picture

At the beginning we started by introducing and emphasizing the "functor of points" or "moduli" perspective in geometry (and throughout mathematics).

This boils down to understanding a "space" (or some other complicated gizmo) by how simpler "test spaces" map into it.

That is, by knowing "families/moduli" of spaces inside it.

The broad context for us was:

∞-cat. of "connective" cdgas over char. 0 field k $=$ $CAlg^{\leq 0}_k$ $= dAff$

$$\begin{tikzcd}
CAlg^{\leq 0}_k \arrow[d, bend right, "H^0"'] \arrow[drr, "\mathfrak{X}"] & & \\
CAlg^{disc}_k \arrow[u, bend right, "\text{inclusion}"'] \arrow[rr, "\mathfrak{X}^{disc}"'] & & \mathcal{S}
\end{tikzcd}$$

$\mathfrak{X}$: derived "moduli" space "DAG"

$\mathcal{S}$: ∞-cat of spaces (or simplicial sets)

category of ordinary comm algs $=$ $CAlg^{disc}_k$

$\mathfrak{X}^{disc}$: underived "moduli" space "AG"

A "rough heuristic", in Lurie's words, is

derived algebraic geometry		underived		
DAG	$=$	AG	$+$	deformation theory
		H^0A		the rest $H^{\neq 0}A$

our course

A good chunk of his new book <u>Spectral AG</u> is devoted to explaining this heuristic

A technical manifestation of this heuristic is the Artin–Lurie representability theorem, which gives criteria under which a functor $\mathcal{X}$ is a derived analog of a Deligne–Mumford stack (and hence "geometric").

$\rightsquigarrow$ A crucial criterion is that for each field point $\eta \in \mathcal{X}(k)$, the completion $\hat{\mathcal{X}}_\eta$ at η is a formal moduli problem whose tangent complex is coconnective (for us $\mathbb{T}_{\hat{\mathcal{X}}_\eta}$ is in nonneg. degrees). Hence you have a feel for the "formal geometry" of geometric derived stacks. //

Returning to the heuristic, I note that it follows from the idea that we induct "down" in coh degree. We did this too, focusing especially on the "dual numbers" $k \oplus k[n] \simeq k[\varepsilon_n]/(\varepsilon_n^2)$

The big picture was

$A \in \mathrm{CAlg}_k^{\le 0}$ w/ aug $A \xrightarrow{\varepsilon} k$
s.t. ① $H^0 A$ is local & fin dim over k
② $H^k A = 0$ for $k \ll 0$
③ $\dim_k H^k A < \infty$ $\forall k$

$$\begin{tikzcd}
\mathrm{Lie}_k \arrow[rr, "\Phi \text{ ③ composite}", "\simeq"'] \arrow[dr, "C^*_{\mathrm{Lie}} \text{ ②}"] \arrow[ddr, bend right, "\text{forget } \oplus \text{ ④}"'] & & \mathrm{FMP} \subseteq \mathrm{Fun}(\mathrm{CAlg}_k^{sm}, S) \arrow[ddl, bend left, "T[-1] \text{ ④ test on dual \#s}"] \\
& (\mathrm{CAlg}_k^{aug})^{op} \arrow[ur, "\mathrm{Spf} \text{ ①}"'] & \\
& \mathrm{Mod}_k &
\end{tikzcd}$$

Yoneda: $\mathrm{Map}_{\mathrm{CAlg}^{aug}}(A, -)$

☆ needed higher categorical machinery to articulate & pursue all this!

The key was a remarkable fact:

Koszul duality, which we understood as an adjoint $\mathfrak{D}$ to C^*_{Lie} & which says that an augmented cdga (i.e., "pointed space") "looks like" $C^*_{\mathrm{Lie}}(\mathfrak{g})$ "near the point"

Hence dg Lie algebras encode formal neighborhoods

☆ ↳ small, convenient models are given by L_∞ algs & constructed via transfer

§1 Duality for modules

The main lecture series focused on that diagram, but it has consequences at a "categorical/module" level, relating two types of "categories of sheaves".

The talks by Dan, Matthew, & Tim touched upon these ideas but presented them in a purely algebraic/rep theoretic way.

Let's sketch Lurie's FMP version.

Big picture of "cats of modules"

$$\begin{tikzcd}
\mathrm{Lie}_k \arrow[d, "\Psi"', "\simeq"] \arrow[rrr, bend left, "(1)\ \mathrm{Rep}"] \arrow[dd, bend right=60, "(3)\ C^*_{\mathrm{Lie}}"'] & & & \\
\mathrm{FMP} \arrow[r] & \mathrm{Fun}(\mathrm{CAlg}^{sm}, S) \arrow[rr, "(4)\ \mathrm{QCoh}"] & & \mathrm{Cat}_\infty \\
\mathrm{CAlg} \arrow[rrru, bend right, "(2)\ \mathrm{Mod}"'] & \mathrm{CAlg}^{sm} \arrow[l] \arrow[u] & &
\end{tikzcd}$$

(1) Rep : $\mathfrak{g} \longmapsto \mathrm{Rep}_{\mathfrak{g}}$ ∞-cat of dg $\mathfrak{g}$ reps (e.g. from model category of $\mathrm{Mod}^{dg}_{U\mathfrak{g}}$)

(2) Mod : $A \mapsto \mathrm{Mod}_A$ dg ∞-cat of dg A-modules

(3) There is an adjunction (even for model categories)

$$\begin{tikzcd} \mathrm{Mod}_{C^*\mathfrak{g}} \arrow[r, shift left, "k \otimes_{C^*\mathfrak{g}}"] \arrow[r, phantom, "\perp"] & \mathrm{Rep}_{\mathfrak{g}} \arrow[l, shift left, "C^*(\mathfrak{g}, -)"] \end{tikzcd}$$

It is almost never an equivalence.

(cf. $\mathfrak{g} = sl_2 \Rightarrow C^* sl_2 \simeq k[\varepsilon]$ $\leftarrow$ deg 3)

<u>But</u> for many $\mathfrak{g}$, $\mathrm{Mod}_{C^*\mathfrak{g}} \longrightarrow \mathrm{Rep}_{\mathfrak{g}}$ is fully faithful.

(4) QCoh is simply the extension of Mod to formal moduli

$$\mathrm{QCoh}(\mathfrak{X}) := \lim_{\substack{\eta \in \mathfrak{X}(A) \\ A \in \mathrm{CAlg}^{\mathrm{sm}}}} \mathrm{Mod}_A$$

<u>Q:</u> How does $\mathrm{Rep}_{\mathfrak{g}}$ compare to $\mathrm{QCoh}(\Psi(\mathfrak{g}))$?

To compare, we should view everything via formal moduli:

$$\begin{tikzcd} & \mathrm{Lie} \arrow[dr, bend left, "\mathrm{Rep}"] & \\ \mathrm{CAlg}^{sm} \arrow[ur, bend left, "\mathfrak{D}"] \arrow[dr, bend right, "\mathcal{O}"'] & & \mathrm{Cat}_\infty \\ & \mathrm{CAlg} \arrow[ur, bend right, "\mathrm{Mod}"'] & \end{tikzcd}$$

<u>Def</u> $\mathrm{QCoh}^!(\mathcal{X}) = \lim\limits_{\substack{\eta \in \mathcal{X}(A) \\ A \in \mathrm{CAlg}^{sm}}} \mathrm{Rep}_{\mathfrak{D}(A)}$

For A small, $A \simeq C^*_{\mathrm{Lie}}(\mathfrak{D}(A))$ & there is a fully faithful embedding

$$\begin{array}{ccc} \mathrm{Mod}_A & \hookrightarrow & \mathrm{Rep}_{\mathfrak{D}(A)} \\ \| & & \| \\ \mathrm{QCoh}(\mathrm{Spf}(A)) & & \mathrm{QCoh}^!(\mathrm{Spf}(A)) \end{array}$$

Hence

<u>Thm</u> For any $\mathfrak{g} \in \mathrm{Lie}_k$, there is a fully faithful embedding

$$\mathrm{QCoh}(\Psi(\mathfrak{g})) \hookrightarrow \underbrace{\mathrm{QCoh}^!(\Psi(\mathfrak{g}))}_{\mathrm{Rep}_{\mathfrak{g}}}$$

Moreover, it induces an equivalence on connective modules

$$\mathrm{QCoh}(\Psi(\mathfrak{g}))^{cn} \xrightarrow{\simeq} \mathrm{QCoh}^!(\Psi(\mathfrak{g}))^{cn}$$

<u>Lesson</u>: $\mathrm{Rep}_{\mathfrak{g}}$ is an "enlargement" of module sheaves on the FMP $\Psi(\mathfrak{g})$ (also called "Ind Coh")

<u>Rmk</u> $\mathrm{QCoh}(\Psi(\mathfrak{g})) \simeq$ filtered modules over the filtered cdga $C^*\mathfrak{g} = \lim C^*\mathfrak{g}/C^*_{\geq n}(\mathfrak{g})$"

Now let me briefly sketch the noncommutative analog, which connects w/ the talks of Dan, Matthew, Tim.

Last week, Claudia explained the following

$$\begin{tikzcd}
\mathrm{Alg}_k^{aug} \arrow[rr, "\Psi_{assoc}", "\simeq"'] \arrow[dr, bend right, "\mathrm{Bar}"'] & & \mathrm{FMP}_{assoc} \subseteq \mathrm{Fun}(\mathrm{Alg}_k^{sm}, S) \\
& (\mathrm{Alg}_k^{aug})^{op} \arrow[ul, bend right, "\mathfrak{D}_{assoc}"'] \arrow[ur, "\mathrm{Spf}"'] &
\end{tikzcd}$$

$A \mapsto \mathrm{Map}_{\mathrm{Alg}_k^{aug}}(A, -)$

Here $\mathrm{Bar}\left(\begin{smallmatrix} A \\ \downarrow\varepsilon \\ k \end{smallmatrix}\right) \simeq \mathbb{R}\mathrm{Hom}_k\left(k \overset{\mathbb{L}}{\underset{A}{\otimes}} k, k\right)$

and the "Koszul dual" is just ~~Bar~~ after taking op's!

<u>Ex</u> $A = U\mathfrak{g}$, $\mathrm{Bar}(U\mathfrak{g}) \simeq C^*_{Lie}(\mathfrak{g})$

Let's talk about Koszul duality for modules, although now we need to keep track of left/right issues.

<u>Def</u> For $\mathcal{X} \in \mathrm{FMP}_{assoc}$,

$$\mathrm{QCoh}_{(L/R)}(\mathcal{X}) = \lim_{\substack{\eta \in \mathcal{X}(A) \\ A \in \mathrm{Alg}^{sm}}} (L/R)\mathrm{Mod}_A$$

There is also a version of $\mathrm{QCoh}^!$

<u>Thm</u> Let $A \in \mathrm{Alg}^{aug}$.
There is a fully faithful embedding

$$\mathrm{QCoh}_{(L/R)}(\Psi_{assoc}(A)) \hookrightarrow \mathrm{QCoh}^!_{(L/R)}(\Psi_{assoc}(A)) \simeq (R/L)\mathrm{Mod}_A$$

On the connective modules, it is an equivalence

$$\mathrm{QCoh}_{(L/R)}\Big(\underset{\text{a isoc}}{\Psi(A)}\Big)^{cn} \simeq (R/L)\mathrm{Mod}_A^{cn}.$$

Note that is similar to what Matt & Toni described: ~~on~~ "connective" complexes, we can identify "dual" modules.

§2 Beyond (non) commutative

We have focused in this course on commutative, Lie, and associative algebras, because they are ubiquitous & familiar.

But these ideas (should) apply to many other kinds of "algebras" (e.g., Gerstenhaber, Leibniz, BV, ...)

$\rightsquigarrow$ one direction: "Koszul operads" & a kind of duality for operads (which are associative algs in a funny category!)

I want to describe briefly a class of algebras that everyone should at least be aware of. They are also ubiquitous & fascinating & they seem to play an increasingly important role in many areas of math

Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category

(e.g., ① $\mathcal{S}$ w/ Cartesian product or

② Mod_k w/ $\otimes$ product)

Fix $n \geq 0$.

<u>Def</u> An <u>E_n algebra</u> in $\mathcal{C}^{\otimes}$ is an object $A \in \mathcal{C}$

and for every $d \geq 0$, a map in $\mathcal{C}^{\otimes}$

$$E_n(d) \otimes_{\Sigma_d} A^{\otimes d} \xrightarrow{m_d} A$$

$\Uparrow$

$\left\{ \begin{array}{l} \text{the space of configurations} \\ \text{of } d \text{ non-overlapping \& labelled boxes inside the} \\ \text{unit box } [0,1]^d \end{array} \right\}$

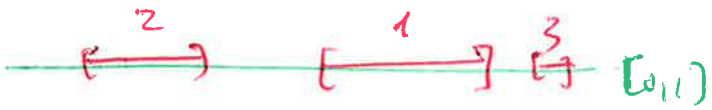
turned into an element of $\mathcal{C}$

(e.g. by C_*^{sing})

and these are "associative" under composition

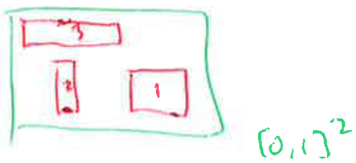
<u>Picture</u>

<u>$n=1$</u>



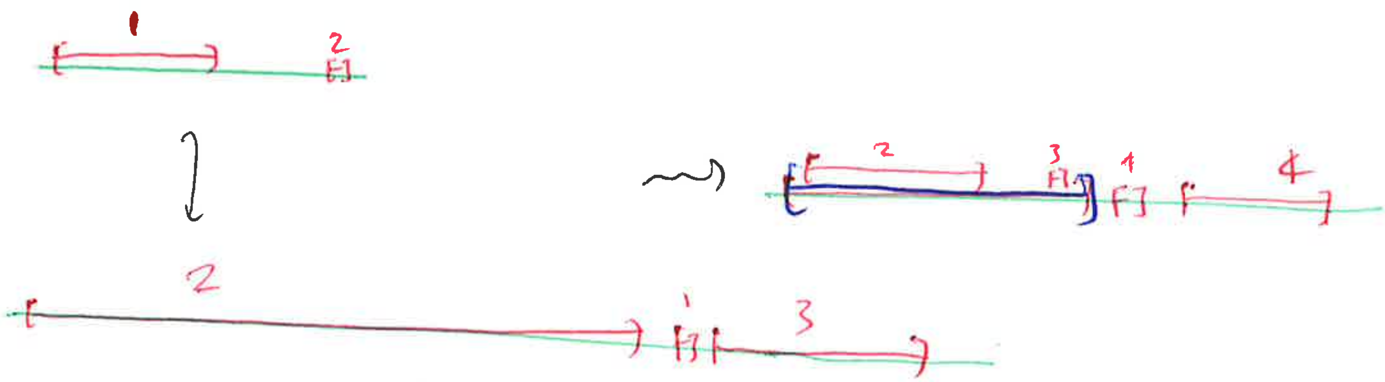
$E_1(d) \simeq d!$ points

<u>$n=2$</u>



$E_2(d) \simeq \mathrm{Conf}^{\mathrm{lab}}_{\mathbb{R}^2}(d)$

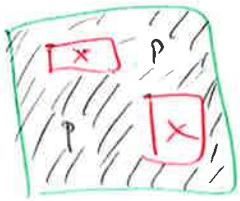
We compose by "insertion"



Fact There is an equiv of ∞-categories: $Alg_{E_1}(\mathcal{C}^{\otimes}) \simeq Alg_{Ass}(\mathcal{C}^{\otimes})$

Ex 1 Let (X, p) be a pointed space.

Let $\Omega^n_p X = \{ f : [0,1]^n \longrightarrow X : f|_{\partial} \equiv p \}$



"n-fold based loops"

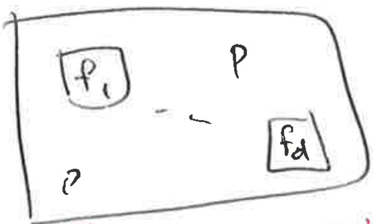
$\pi_0(\Omega^n_p X) = \pi_n(X, p)$

Claim $\Omega^n X$ is an E_n-algebra in $\mathcal{S}$.

It should be clear how to define maps:

given $(f_1, \dots, f_d) \in (\Omega^n X)^d$ &  $\in E_n(d)$

just insert



This is a "lift" of the (ab) group str on $\pi_n X$ III

~~Taking Π_0, we recover Π_n~~

Ex 2 These also appear in purely algebraic contexts as well, which should be somewhat surprising

Thm (proved by lots of people: McClure-Smith, Tamarkin, Kontsevich-Soib., Voronov...) "Deligne's conjecture"

For A an associative alg (or dg alg), the Hochschild cochain complex

$$\mathrm{Hoch}^*(A,A)$$ ← "deformations as algebra"

has a canonical (up to homotopy!) E_2-algebra structure.

This result says something kind of remarkable: in trying to understand "families of algebras" (eg deformations) leads to a 2-dimensional version of algebra (E_2 algs)

There is, in fact, a "tower" of higher algebras

$$E_1 \subset E_2 \subset \cdots \subset E_\infty$$

E_1 = assoc; less commutative; E_∞ = commutative

& we have barely begun exploring

IV